ON THE QUASIANALYTIC WAVE-FRONT SET

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ABSTRACT. We introduce s-wave-front sets for ultra-distributions of Beurling and Roumieu types, where $s \in (0,1]$. The assumptions on s implies that the s-wave-front sets contain the analytic wave-front sets. Hence, in some sense they give information about stronger regularity compared to what can be obtained from the analytic wave-front sets. We also establish common properties for such wave-front sets, e. g. micro-local properties for partial differential operators. We also relate these wave-front sets to s-singular supports.

0. INTRODUCTION

In the paper we introduce a family of wav-front sets on Gevrey distributions, which contains the analytic wave-front sets. (See e. g. [2].) For any $s > 0$, $t > 1$ and $u \in \mathcal{D}_t'(X)$, we define two types of wave-front sets for u

$$\overline{WF}_s(u) \quad \text{and} \quad \overline{\overline{WF}}_s(u) \tag{0.1}$$

for u.

Here s is the parameter in the Gevrey space $\mathcal{E}_s$. Roughly speaking, the wave-front sets in (0.1) explains where u locally fails to belong to $\mathcal{E}_s$, as well as the propagations of these singularities.

If $s \geq 1$, then the wave-front sets in (0.1) coincide, and are equal to certain wave-front sets of the form $WF_L(u)$ in Section 8.4 in [2]. In particular, if $s = 1$, then they agree with the (real-)analytic wave-front set, $WF_A(u)$ (cf. e. g. [2]).

The wave-front sets in (0.1) decrease with the parameter s. In particular, WF_A is contained in these wave-front sets when $s < 1$, and strict inclusions are attained for certain u. Here we note that for any wave-front set $WF_*(u)$ in the literature, it always seems that

$$WF_*(u) \subseteq WF_A(u).$$

That is, roughly speaking, in the literature there are no wave-front set which detect heavier singularities than singularities with respect to local analyticity. With this respect, due to the inclusion

$$WF_A(u) \subseteq \overline{WF}_s(u) \subseteq \overline{\overline{WF}}_s(u), \quad s \leq 1,$$

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the wave-front sets in (0.1) inform about higher regularity properties compared to real-analyticity, when $s \le 1$.

We also establish basic properties for these wave-front sets. For example, we prove

$$\pi_1(\overline{WF_s}(u)) \subset \operatorname{singsupp}_s u \subset \pi_1(\overline{\overline{WF_s}}(u)), \tag{0.2}$$

when π_1 is the projection $\pi_1(x,\xi) = x$ from $\in \mathbf{R}^{2d}$ to $\mathbf{R}^d$. In particular, equalities are attained in (0.2) when $s \ge 1$. In the end we also show that the wave-front sets decreases when differential operators are applied on the distributions Consequently, the wave-front sets here can be applied on problems involving partial differential equations.

We also show multiplication and tensor product properties for the wave-front sets.

1. Quasi-analytic wave front of distributions

We are interested in Gevrey type regularities, related to the sequence $(N!^s)_N$, $s \in (0,1]$, for Schwartz distributions as well as for ultradistributions of Beurling and Roumieu type. This is done in the same spirit as it is done for the analytic classes in [2, Chapter VIII].

Let $X \subseteq \mathbf{R}^d$ be open. Then $\mathcal{D}'_t(X)$ and $\mathcal{D}'_{\{t\}}(X)$ are the sets of ultradistributions on X of Beurling respectively Roumieu type to the order $t > 1$. We set

$$\mathcal{D}'_{\mho}(X) \equiv \cup_{t>1} \mathcal{D}'_t(X), \tag{1.1}$$

and equip $\mathcal{D}'_{\mho}(X)$ with the inductive limit topology. Note that the union in (1.1) remains the same if $\mathcal{D}'_t$ is replaced by $\mathcal{D}'_{\{t\}}$.

We also let $\mathcal{E}'_t(X)$, $\mathcal{E}_{\{t\}}(X)$ and $\mathcal{E}'_{\mho}(X)$ be all elements in $\mathcal{D}'_t(X)$, $\mathcal{D}_{\{t\}}(X)$ and $\mathcal{D}'_{\mho}(X)$, respectively, with compact supports.

We recall that $\Gamma \subseteq \mathbf{R}^d \setminus \{0\}$ is an open cone, if and only if $t\xi \in \Gamma$ when $t > 0$ and $\xi \in \Gamma$. A cone $F \subseteq$ is called closed (in $\mathbf{R}^d \setminus \{0\}$) if its complement in $\mathbf{R}^d \setminus \{0\}$ is an open cone.

An essential part of the investigations concerns sequences in the following definition.

Definition 1.1. Let $K \subseteq \mathbf{R}^d$ be compact, $s \in (0,1]$ and $t > 1$. A sequence $(\chi_N)_{N\in\mathbf{N}}$ in $C_0^\infty(\mathbf{R}^d)$ is called *$\mathcal{D}_t$-feasible with respect to K and s*, if $\operatorname{supp}(\chi_N) \subset K$ for all $N \in \mathbb{N}$, and for every $h > 0$, there is a constant $C_h > 0$ such that

$$\sup_{x\in\mathbf{R}^d} |\chi_N^{(\alpha+\beta)}(x)| \le C_h^{|\alpha|+1} h^{|\beta|} \beta!^t \lfloor N^s \rfloor^{|\alpha|}, \quad |\alpha| \le \lfloor N^s \rfloor,\ N \in \mathbf{N}, \beta \in \mathbf{N}^d. \tag{1.2}$$

For future references we note that if (χ_N) and $(\tilde\chi_N)$ are $\mathcal{D}_t$-feasible with respect to K and s, then the same is true for $(\chi_N \tilde\chi_N)$.

Remark 1.2. If $K \subseteq \mathbf{R}^d$ be compact, $s \in (0,1]$, $t > 1$ and (χ_N) is $\mathcal{D}_t$-feasible with respect to K and s, then for every $c > 0$, there is a constant C such that

$$|\widehat{\chi}_N(\xi)| \le C^{|\alpha|+1} \lfloor N^s \rfloor^{|\alpha|} e^{-c|\xi|^{1/t}} \langle \xi \rangle^{-|\alpha|}, \quad |\alpha| \le \lfloor N^s \rfloor,\ N \in \mathbf{N}. \tag{1.3}$$

This follows by similar arguments which is used in e. g. [1].

Lemma 1.3. *Let $t > 1$, $s \in (0,1]$, $X \subseteq \mathbf{R}^d$ be open, $K \subseteq X$ be compact, and set $K_r = \{x : d(x,K) \le r\}$ when $r > 0$. Then there exists a $\mathcal{D}_t$-feasible sequence with respect to K_r and s such that $\chi_N \equiv 1$ in K for every $N \in \mathbf{N}$.*

Before the proof we have the following remark concerning special cases of Lemma 1.3. Here $B_r(x)$ denotes the open ball centered at $x \in \mathbf{R}^d$ and radius $r > 0$.

Remark 1.4. Let $t > 1$, $r > 0$, $X \subseteq \mathbf{R}^d$ be open and $K \subseteq X$ be compact. Then it follows from Lemma 1.3 that the following is true:

(1) There exists a sequence $(\tilde{\chi}_N)_{N\in\mathbb{N}}$ in $C_0^\infty(X)$ such that $\tilde{\chi}_N \equiv 1$ in a neighborhood of K, supp $(\tilde{\chi}_N) \subset K_r$ for all $N \in \mathbf{N}$ and the derivatives satisfy

$$\sup_{x\in\mathbf{R}^d} |\tilde{\chi}_N^{(\alpha)}(x)| \le C^{|\alpha|} \lfloor N^s \rfloor^{|\alpha|}, \quad |\alpha| \le \lfloor N^s \rfloor,\ N \in \mathbf{N}, \tag{1.4}$$

where $\lfloor x \rfloor$ is the integer part of $x > 0$.

(2) Let $r_0 > 0$, $s \in (0,1]$ and let $x_0 \in \mathbf{R}^d$ be fixed. Then there exists a $\mathcal{D}_t$-feasible sequence with respect to $B_{2r_0}(x_0)$ and s such that $\chi_N \equiv 1$ in $B_{r_0}(x_0)$ for every $N \in \mathbf{N}$.

Proof. We may assume that r has been chosen such that $K_r \subseteq X$. In the first step we prove the special cases (1) and (2) in Remark 1.4. The asertion (1) follows from [2, Theorems 1.4.1, 1.4.2], after the change of indices. Instead of N in Hörmander's construction we put $\lfloor N^s \rfloor$.

(2) Let $K = \{x_0\}$ in (1), and let $\theta \in \mathcal{D}_t$ be non-negative, supported in $B_{r/4}(0)$ and with integral equal to one. This is possible because $t > 1$. By (1) there is a sequence $(\tilde{\chi}_N)$ of smooth functions such that $\operatorname{supp} \tilde{\chi}_N \subseteq B_{7r/4}(x_0)$, $\tilde{\chi}_N = 1$ on $B_{5r/4}(x_0)$ and such that (1.4) holds. The result now follows by letting $\chi_N = \theta * \tilde{\chi}_N$.

The general case now follows by convolving the characteristic function to K_{r_0} for some appropriate r_0 with an element in (2) after r has been modified in a suitable way. □

Next we establish necessary and sufficient conditions for distributions such that they should locally belong to $\mathcal{E}_s$. The simple inequality

$$\lfloor N^s \rfloor^N \le N^{sN} \le C^{N+1} N!^s, \text{ for some C } > 0 \tag{1.5}$$

is important in these considerations.

Proposition 1.5. *Let $X \subseteq \mathbf{R}^d$ and $U \subseteq X$ be open, $u \in \mathcal{D}'_{\mho}(X)$ and $(u_N)_{N\in\mathbf{N}}$ be a bounded sequence in $\mathcal{E}'_{\mho}(X)$ such that $u_N = u$ on U. If*

$$|\widehat{u}_N(\xi)| \le \frac{C^{N+1}N!^s}{|\xi|^N}, \quad N \in \mathbf{N}, \tag{1.6}$$

for some $C > 0$, then $u \in \mathcal{E}_s(U)$. That is

$$\sup_{x\in U} |D^{(\alpha)}u(x)| \le C^{|\alpha|+1}|\alpha|!^s, \quad \forall \alpha \in \mathbf{N}^d. \tag{1.7}$$

Remark 1.6. Let $t > 1$. Then it follows from the definitions that Proposition 1.5 remains true if $\mathcal{D}'_{\mho}$ and $\mathcal{E}'_{\mho}$ are replaced by $\mathcal{D}'_t$ and $\mathcal{E}'_t$ respectively, or if they are replaced by $\mathcal{D}'_{\{t\}}$ and $\mathcal{E}'_{\{t\}}$ respectively.

Proof. Since $\mathcal{D}'_{\mho}$ is equipped by the inductive limit topology, it follows that $u \in \mathcal{D}'_t$ and (u_N) is bounded in $\mathcal{E}'_t$ for some $t > 1$, and

$$|\widehat{u}_N(\xi)| \le Ce^{c|\xi|^{1/t}}, \quad \xi \in \mathbf{R}^d,\ N \in \mathbf{N}, \tag{1.8}$$

for some positive constants C and c which do not depend on N.

Let $C_1 > C$. From (1.6), Fourier's inversion formula and the fact that $u_N = u$ on U, we obtain

$$\begin{aligned}(C_1^{|\alpha|+1}|\alpha|!^s)^{-1}|D^\alpha u(x)| &\le (C_1^{|\alpha|+1}|\alpha|!^s)^{-1}\left|\int \xi^\alpha \widehat{u}_N(\xi)e^{i\langle x,\xi\rangle}\, d\xi\right| \\ &\le I_1 + I_2, \quad x \in U,\end{aligned}$$

where

$$I_1 = (C_1^{|\alpha|+1}|\alpha|!^s)^{-1}\left|\int_{|\xi|\le 1} \xi^\alpha \widehat{u}_N(\xi)e^{i\langle x,\xi\rangle}\, d\xi\right|$$

and

$$I_2 = (C_1^{|\alpha|+1}|\alpha|!^s)^{-1}\left|\int_{|\xi|\ge 1} \xi^\alpha \widehat{u}_N(\xi)e^{i\langle x,\xi\rangle}\, d\xi\right|.$$

By (1.8) we get

$$I_1 \le (C_1^{|\alpha|+1}|\alpha|!^s)^{-1}\int_{|\xi|\le 1} e^{c|\xi|^{1/t}}\, d\xi \le C_2,$$

for some $C_2 < \infty$ which is independent of α. In order to estimate I_2 we choose $N = |\alpha| + d + 1$. Then (1.6) and the fact that $C_1 > C$ give

$$I_2 \le (C_1^{|\alpha|+1}|\alpha|!^s)^{-1}C^{N+1}N!^s\int_{|\xi|\ge 1} |\xi|^{|\alpha|-N}\, d\xi \le C_2,$$

for some $C_2 > 0$ which is independent of α. This gives (1.7), and the proof is complete. $\square$

Proposition 1.7. *Let $X \subseteq \mathbf{R}^d$ open, U be open with compact closure contained in X, and $u \in \mathcal{E}_s(U)$. Then there exists a bounded sequence $(u_N)_{N\in\mathbf{N}}$ in $\mathcal{E}'(\mathbf{R}^d)$ such that $u_N = u$ on U and*

$$|\widehat{u}_N(\xi)| \le C^{N+1}\frac{\lfloor N^s\rfloor !}{|\xi|^{\lfloor N^s\rfloor}}, \quad N \in \mathbf{N} \tag{1.9}$$

for some $C > 0$.

Proof. Let K be compact such that $U \subseteq K \subseteq X$, and let $r > 0$ be such that $K_r \subseteq X$. Also let (χ_N) be the sequence in Lemma 1.3 and define $u_N = \chi_N u$, $N \in \mathbf{N}$. Then $u_N = u$ on U and $(u_N)_{N\in\mathbf{N}}$ is a bounded sequence in $\mathcal{E}'(\mathbf{R}^d)$.

Let $|\alpha| \le \lfloor N^s\rfloor$ and $x \in U$. Then Leibnitz rule gives

$$|D^\alpha u_N(x)| \le \sum_{|\beta|\le|\alpha|}\binom{\alpha}{\beta}|D^{\alpha-\beta}\chi_N(x)||D^\beta u(x)|$$

$$\le \sum_{|\beta|\le|\alpha|}\binom{\alpha}{\beta}C^{1+|\alpha-\beta|+|\beta|}\lfloor N^s\rfloor^{|\alpha-\beta|}|\beta|!^s.$$

Since $\beta \le \alpha$ and $|\alpha-\beta| + |\beta| = |\alpha|$, it follows

$$\lfloor N^s\rfloor^{|\alpha-\beta|}|\beta|^{s|\beta|} \le \lfloor N^s\rfloor^{|\alpha-\beta|}|\beta|^{|\beta|} \le \lfloor N^s\rfloor^{|\alpha|}.$$

This implies

$$|D^\alpha u_N(x)| \le C^{N+1}\lfloor N^s\rfloor^{|\alpha|}. \tag{1.10}$$

Since there is a compact set $K \subseteq X$ such that $\operatorname{supp} u_N \subseteq K$ for every N, (1.9) follows by applying the Fourier transform on (1.10). □

Now, we give the definitions of the wave front $\overline{WF}$ and of the wave front $\overline{\overline{WF}}$ of a distribution. If $s = 1$, then these two sets equals. For $s < 1$ these two sets bound the microlocal regualrity at $(x_0,\xi_0) \in \mathbf{R}^d \times \mathbf{R}^d \setminus \{0\}$

Definition 1.8. Let $s \in (0,1]$, $X \subseteq \mathbf{R}^d$ be open, $u \in \mathcal{D}'_{\mho}(X)$ and $(x_0,\xi_0) \in \mathbf{R}^d \times \mathbf{R}^d \setminus \{0\}$. Then

- $(x_0,\xi_0) \notin \overline{WF_s}(u)$ if there exists a conical neighborhood $U\times\Gamma \subset X \times (\mathbf{R}^d \setminus \{0\})$ of (x_0,ξ_0) and a bounded sequence $u_N \in \mathcal{E}'_{\mho}(X)$ so that $U_N = u$ on U and

$$|\widehat{u}_N(\xi)| \le \frac{C^{N+1}\lfloor N^s\rfloor !}{|\xi|^{\lfloor N^s\rfloor}}, \quad N \in \mathbf{N},\ \xi \in \Gamma. \tag{1.11}$$

- $(x_0,\xi_0) \notin \overline{\overline{WF_s}}(u)$ if there exists a conical neighborhood $U\times\Gamma \subset X \times \mathbf{R}^d \setminus \{0\}$ of (x_0,ξ_0) and a bounded sequence $u_N \in \mathcal{E}'_{\mho}(\mathbf{R}^d)$ so that $u_N = u$ in U and

$$|\widehat{u}_N(\xi)| \le \frac{C^{N+1}N!^s}{|\xi|^N}, \quad N \in \mathbf{N},\ \xi \in \Gamma. \tag{1.12}$$

Obviously, if $u \in \mathcal{D}'_{\mho}$ and $s \in (0,1]$, then

$$\overline{WF_s}(u) \subseteq \overline{\overline{WF_s}}(u),$$

with equality when $s = 1$.

Next, we compare the projections of these sets with the singular support with respect to $\mathcal{E}_s$. We need the following lemma.

Lemma 1.9. *Let $s \in (0,1]$, $t > 1$, $X \subseteq \mathbf{R}^d$ be open, $u \in \mathcal{D}'_t(X)$, $K \subseteq X$ be compact, F be a closed cone, and let (χ_N) be $\mathcal{D}_t$-feasible with respect to K and s. Then the following is true:*

(1) *$(\chi_N u)$ is a bounded sequence in $\mathcal{E}'_t$;*

(2) *if $\overline{WF_s}(u) \cap (K \times F) = \emptyset$, then for some $C > 0$,*

$$|\widehat{\chi_N u}(\xi)| \le C^{N+1} \frac{\lfloor N^s \rfloor !}{|\xi|^{\lfloor N^s \rfloor}}, \quad N \in \mathbf{N},\ \xi \in F; \tag{1.13}$$

(3) *if $\overline{\overline{WF}}_s(u) \cap (K \times F) = \emptyset$, then for some $C > 0$,*

$$|\widehat{\chi_N u}(\xi)| \le C^{N+1} \frac{N!^s}{|\xi|^N}, \quad N \in \mathbf{N},\ \xi \in F. \tag{1.14}$$

Proof. We only prove (1) and (2). The assertion (3) follows by similar arguments as for (2) and is left for the reader.

It is clear that $\chi_N u$ is bounded in $\mathcal{E}'_t(\mathbf{R}^d)$ since χ_N is bounded in $\mathcal{D}_t(\mathbf{R}^d)$. We will use the same ideas as for the proof of [2, Lemma 8.4.4] when proving (2).

Let $(x_0, \xi_0) \in K \times F$ be fixed, u_N, U and Γ be as in Definition 1.8 (1), and choose $r_0 = r_{x_0,\xi_0} > 0$ such that $B_{r_0}(x_0) \subseteq U$. Also let (χ_N) be $\mathcal{D}_t$-feasible with respect to $B_{r_0}(x_0)$ and s, and let Γ_0 be an open conical neighbourhood of ξ_0 with closure contained in Γ. Then $\chi_N u = \chi_N u_N$, and (1.8) holds for some constants c and C. Moreover, (1.6) holds true when $\xi \in \Gamma$.

Let $\Omega_1 = B_1(0)$, $\Omega_2 = \Gamma \setminus B_1(0)$, and let Ω_3 be the complement of Γ. Then

$$|\widehat{\chi_N u}(\xi)| = |\widehat{\chi_N u_N}(\xi)| \lesssim I_1 + I_2 + I_3, \tag{1.15}$$

where

$$I_j = \int_{\Omega_j} |\widehat{\chi}_N(\xi - \eta)|\, |u_N(\eta)|\, d\eta.$$

By letting $|\alpha| = \lfloor N^s \rfloor$ in (1.3), (1.8) gives

$$I_1 \lesssim C^{N+1} \lfloor N^s \rfloor^{\lfloor N^s \rfloor} \int_{|\eta| \le 1} e^{-c|\xi - \eta|^{1/t}} \langle \xi - \eta \rangle^{-\lfloor N^s \rfloor}\, d\eta$$

$$\lesssim C_1^{N+1} \lfloor N^s \rfloor ! e^{-c|\xi|^{1/t}}. \tag{1.16}$$

By letting $\alpha = 0$ in (1.3), (1.13) gives

$$I_2 \lesssim C^{N+1} \lfloor N^s \rfloor ! \int_{\mathbf{R}^d} e^{-c|\xi-\eta|^{1/t}} \langle \eta \rangle^{-\lfloor N^s \rfloor} \, d\eta$$

$$\lesssim C^{N+1} \lfloor N^s \rfloor ! \langle \xi \rangle^{-\lfloor N^s \rfloor} \int e^{-c|\xi-\eta|^{1/t}/2} \, d\eta \asymp C^{N+1} \lfloor N^s \rfloor ! \langle \xi \rangle^{-\lfloor N^s \rfloor}. \qquad (1.17)$$

Finally, in order to establish an estimate for I_2, we note that for some constant $c_0 > 0$ we have

$$|\xi - \eta|^{1/t} \ge c_0(|\xi|^{1/t} + |\eta|^{1/t}),$$

when $\eta \in \Omega_3$ and $\xi \in \Gamma_0$, which implies that

$$e^{-c|\xi-\eta|^{1/t}} \le e^{-c_0 c|\xi|^{1/t}} e^{-c_0 c|\eta|^{1/t}}.$$

Hence, if $c_1 > 0$ is fixed, then by choosing $\alpha = 0$, and c in (1.3) large enough, (1.8) gives

$$|\widehat{\chi}_N(\xi - \eta)| \, |u_N(\eta)| \lesssim e^{-c_1|\xi|^{1/t}} e^{-c_1|\eta|^{1/t}}, \quad \xi \in \Gamma_0, \ \eta \in \Omega_3$$

for some constant $c_1 > 0$. By integrating the last inequality over $\eta \in \Omega_3$, we get

$$I_3 \lesssim e^{-c|\xi|^{1/t}}, \qquad (1.18)$$

for any choice of $c > 0$. By combining (1.15)–(1.18), the estimate (1.13) now follows in this case, and with F replaced by Γ_0.

For general F we note that the intersection of F with the unit sphere is compact. Hence we may choose finite numbers of balls and cones, $B_{x_0}(r_{x_0,\xi_j})$ and Γ_j, $j = 1, \dots, n$ such that

$$\Gamma_{x_0} \equiv \cup_{j=1}^n \Gamma_j$$

covers F. Furthermore, if (χ_N) are chosen such that their supports are contained in the intersection, B_{x_0}, of these balls, then (1.13) holds.

Finally, since K is compact, we may cover K with by finite number of open balls B_{x_k}, $k = 1, \dots, m$, and choose appropriate functions $\chi_{N,k} \in C_0^\infty(B_{x_k})$ such that $\sum \chi_{N,k} = 1$ in K and $\chi_{N,k}$ satisfy (1.2) for $k = 1, \dots, m$. Then $\chi_{N,k}\chi_N$ also satisfies (1.2), with some other constant. We conclude that (1.13) holds with χ_N replaced by $\chi_{N,k}\chi_N$. Since $\sum \chi_{N,k}\chi_N = 1$, the result follows. □

The following result links the s-singular support with our wave-front sets.

Theorem 1.10. *Let $s \in (0,1]$ and $u \in \mathcal{D}'_{\mho}(X)$. Then* (0.2) *holds.*

Proof. Let $t > 1$ be chosen such that $u \in \mathcal{D}'_t(X)$. If $x_0 \notin \operatorname{singsupp}_s u$ then Proposition 1.7 implies (1.9) and we conclude that $(x_0, \xi_0) \notin \overline{W}F_s(u)$ for any $\xi_0 \in \mathbf{R}^d \setminus 0$.

Conversely, if $(x_0, \xi_0) \notin \overline{\overline{\overline{WF}}}_s(u)$ for all $\xi_0 \in \mathbf{R}^d \setminus 0$, then we can choose a compact neighborhood K of x_0 such that $\overline{\overline{WF}}_s(u) \cap (K \times \mathbf{R}^d) =$

$\emptyset$. By Lemma 1.9 there is a a bounded sequence $\{u_N\}$ in $\mathcal{E}_t'(\mathbf{R}^d)$ such that $u_N = u$ on some open set U and

$$|\widehat{u}_N(\xi)| \le \frac{C^{N+1}N!^s}{|\xi|^N}, \quad N \in \mathbf{N}, \tag{1.19}$$

for some $C > 0$. By Proposition 1.5 we conclude that $u \in \mathcal{E}_s(U)$, that is that $x_0 \notin \operatorname{singsupp}_s u$. $\square$

We have now the following result. Here $\operatorname{Char}(P)$ stands for the set of characteristic points to the partial differential operator P (cf. Chapter XVIII in [2] for strict the definition).

Theorem 1.11. *Let $s \in (0,1]$, $t > 1$, $X \subseteq \mathbf{R}^d$, and let $u \in \mathcal{D}_t'(X)$. Also let $WF_s(u)$ be any of the sets $\overline{WF}_s(u)$ and $\overline{\overline{WF}}_s(u)$. Then the following is true:*

1. *If $0 < s_1 < s_2 \le 1$, then $WF_A(u) \subset WF_{s_2}(u) \subset WF_{s_1}(u)$;*
2. *If $\phi \in \mathcal{E}_s(X)$, then $WF_s(\phi u) \subset WF_s(u)$;*
3. *Let $P(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$, where $a_\alpha \in \mathcal{E}_s(X)$ for every α. Then*

$$WF_s(P(x,D)u) \subseteq WF_s(u) \subseteq WF_s(P(x,D)u) \cup \operatorname{Char}(P). \tag{1.20}$$

Proof. All statements, except the second inclusion in (1.20) are straightforward consequences of the definitions and previous results. The details are left for the reader.

The second inclusion in (1.20) follows by straight-forward modifications of the proof of Theorem 8.6.1 in [2]. Again we leave the details for the reader. $\square$

Example 1.12. Let $u \in \mathcal{D}_\mho'(X)$ be harmonic. Then it follows from Theorem 1.11 that $u \in \mathcal{E}_s(X)$, for every s.

In fact, let $s > 0$. Since $\operatorname{Char}(\Delta) = \emptyset$ and $\Delta u = 0 \in \mathcal{E}_s$, it follows from (1.20) that $WF_s(u) = \emptyset$. The assertion now follows from Theorems 1.10 and 1.11(1).

2. Computational rules for the s wave-front sets

In this section we present computational rules for the s wave-front set. In most of the cases we exclude the proofs, or present only some ideas to the proof, since the proofs are similar to corresponding results in [2].

We start with the following result on tensor products and multiplication of distributions. Here and in what follows we use $WF_s(u)$ to denote any of the sets $\overline{WF}_s(u)$ and $\overline{\overline{WF}}_s(u)$.

Proposition 2.1. *Let $s \in (0,1]$, $X \in \mathbf{R}^{d_1}$ and $Y \in \mathbf{R}^{d_2}$ be open, $u \in \mathcal{D}'_{\mho}(X)$ and let $v \in \mathcal{D}'_{\mho}(Y)$. Then $u \otimes v \in \mathcal{D}'_{\mho}(X \times Y)$, and*

$$
\begin{aligned}
WF_s(u \otimes v) \subseteq &\big(WF_s(u) \times WF_s(v)\big)\\
&\cup \big((\operatorname{supp} u \times \{0\}) \times WF_s(v)\big) \cup \big(WF_s(u) \times (\operatorname{supp} v \times \{0\})\big);
\end{aligned}
$$

Iff is a linear map from X to Y, then Theorem 8.2.4 in [2] remains true, after

$$
u \in \mathscr{D}'(X), \quad WF(u) \quad \text{and} \quad WF(f^*u)
$$

have been replaced by

$$
u \in \mathcal{D}'_t(X), \quad \overline{\overline{WF}}_s(u) \quad \text{and} \quad \overline{\overline{WF}}_s(f^*u),
$$

respectively, provided $0 < s \le t$ and $t > 1$. In particular,

$$
\begin{aligned}
&\overline{\overline{WF}}_s(f^*u) \subseteq f^*\overline{\overline{WF}}_s(u),\\
&\qquad\qquad \text{when} \quad N_f \cap \overline{\overline{WF}}_s(u) = \emptyset \text{ and } u \in \mathcal{D}'_t(X). \qquad (2.1)
\end{aligned}
$$

We refer to the proofs of Theorems 8.2.4 and 8.5.1 in [2] for the arguments.

By using (2.1) and Proposition 2.1 we get the following.

Proposition 2.2. *Let $f(x) = (x,x)$ when $x \in X$, $1 < t$, $0 < s \le t$, and let $u, v \in \mathcal{D}'_t(X)$ be such that*

$$
(x,\xi) \notin \overline{\overline{WF}}_s(u) \quad \textit{when} \quad (x,\xi) \in \overline{\overline{WF}}_s(v).
$$

Then the product $u \cdot v \equiv f^*(u \otimes v)$ is well-defined and belongs to $\mathcal{D}'_t(X)$. Furthermore,

$$
\begin{aligned}
\overline{\overline{WF}}_s(u \cdot v) \subseteq \{\, (x,\xi+\eta)\,;\, &(x,\xi) \in \overline{\overline{WF}}_s(u) \ \textit{or}\ \xi = 0,\ \textit{and}\\
&(x,\eta) \in \overline{\overline{WF}}_s(v) \ \textit{or}\ \eta = 0 \,\}.
\end{aligned}
$$

Finally we remark that Theorems 8.2.12–8.2.14, 8.5.4′ and 8.5.5 in [2], on wave-front results on distribution kernels in, are true also when the wave-front sets and distribution spaces have been replaced by s-wave-front sets and $\mathcal{D}'_t$, provided $1 < t$ and $0 < s \le t$. We leave the verifications for the reader. By using these results we obtain

$$
\overline{\overline{WF}}_s(u * v) \subseteq \{\, (x+y,\xi)\,;\, (x,\xi) \in \overline{\overline{WF}}_s(u) \text{ and } (y,\xi) \in \overline{\overline{WF}}_s(v) \,\},
$$

when $u \in \mathcal{D}'_t(\mathbf{R}^d)$ and $v \in \mathcal{E}'_t(\mathbf{R}^d)$.

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